A PARAMETERIZATION OF THE LARGE-SCALE TRANSIENT EDDY FLUX OF RELATIVE ANGULAR MOMENTUM

BARRY SALTZMAN[1] and ANANDU D. VERNEKAR

The Travelers Research Center, Inc., Hartford, Conn.

ABSTRACT

The barotropic vorticity equation and zonal kinetic energy equation are used to derive a formula expressing the transport of relative angular momentum by large-scale transient eddies in terms of other mean zonally averaged variables. To test the formula, computations are made using the observed winter and summer zonal average conditions. Generally good agreement is found between the calculated and observed momentum transport.

1. INTRODUCTION

In order to close the system of equations governing the time-averaged (i.e., mean) and zonally averaged (i.e., symmetric) state of the atmosphere, it is necessary to represent the mean zonally averaged transient eddy transport of relative angular momentum in terms of the mean zonally averaged dependent variables (cf., e.g., Saltzman [3], Smagorinsky [7], Williams and Davies [9]). We shall derive one such representation based on the barotropic vorticity equation and the conservation equation for zonal kinetic energy. The representation is capable of accounting for the main features of the observed flux of momentum for winter and summer average conditions.

First, let us define the following symbols,

λ=longitude
ϕ=latitude
p=pressure
t=time
$u=a\cos\phi\, d\lambda/dt$ (eastward wind speed)
$v=a\, d\phi/dt$ (northward wind speed)
$\omega=dp/dt$
z=geopotential
g=acceleration of gravity
$\Phi=gz$
a=radius of the earth
Ω=rate of rotation of the earth
$f=2\Omega\sin\phi$ (Coriolis parameter)
f^*=value of f at standard latitude (e.g., 45°)
$\psi=\Phi/f^*$
$\beta=df/ad\phi$
x=eastward viscous force per unit mass,

and the following modes of averaging, for any variable such as ψ,

$\bar{\psi}=\frac{1}{\tau}\int_{-\tau/2}^{\tau/2}\psi dt$ (mean over time interval τ)

$\psi'=\psi-\bar{\psi}$ (transient departure)

$\psi_0=\frac{1}{2\pi}\int_0^{2\pi}\bar{\psi}d\lambda$ (mean zonally averaged variable)

$\psi_1=\bar{\psi}-\psi_0$ ("standing eddy" departure)

$\{\psi\}=\frac{1}{2p_s}\int_0^{p_s}\int_{-\pi/2}^{\pi/2}\psi\cos\phi\, d\phi\, dp$ (meridional and vertical average over the globe).

2. DERIVATION

We start by assuming that the traveling perturbations induced baroclinically in the atmosphere can be represented in the form

$$\begin{aligned}\psi^*(\lambda,\phi,p,\epsilon;t)&=\psi-\psi_0\\&=\Psi(\phi,p)\cos n[\lambda-\Lambda(t)-\mu(\phi)\epsilon],\end{aligned}\qquad(1)$$

where n is a wave number characteristic of the unstable waves; t is the time of inception of the wave; ϵ is the time elapsed after inception; $\Lambda(t)$ is the initial phase angle of the wave, which is uniform with latitude; and μ is the angular phase speed of the wave which may vary with latitude. If we further assume that there is an equal probability that Λ will take on any value between 0 and 2π (i.e., that the phase of the developing baroclinic wave is arbitrary),[2] we can expect that for a time interval τ of sufficient length (e.g., several months) we have,

$$\bar{\psi}^*=\frac{1}{\tau}\int_{-\tau/2}^{\tau/2}\psi dt\rightarrow 0,$$

or

$$\psi^*\rightarrow\psi'.$$

Since it is our intention to parameterize the effects of the transient baroclinic waves, we should also require that the interval τ be chosen short enough to exclude the main part of the variance associated with the annual cycle. A 6-mo., solstice-centered averaging period such as studied by Buch [1] would seem to be reasonable. In this case, the monsoonal land-sea effects associated with the annual cycle of heating would be represented mainly by the standing-eddy component ψ_1.

The mean zonally averaged transport of momentum by these transient waves at some characteristic time after inception $\epsilon=T$, computed from the nondivergent components of the wind,

$$u'=-\frac{\partial\psi'}{a\partial\phi}$$

$$v'=\frac{\partial\psi'}{a\cos\phi\partial\lambda},$$

[1] New address: Department of Geology and Geophysics, Yale University, New Haven, Conn.

[2] This assumption is not fully satisfied in the atmosphere because the continent-ocean structure tends to favor certain locations for cyclogenesis.

is given by

$$(u'v')_0 = T\sigma_v^2 \cos\phi \frac{d\mu}{d\phi}, \tag{2}$$

where

$$\sigma_v^2 = (v'^2)_0 = \frac{n^2}{2a^2\cos^2\phi}\Psi^2.$$

This kinematical relationship is the basis for the parameterization, showing qualitatively that the momentum flux is proportional to the amplitude of the waves (as measured by the variance of the meridional wind, σ_v^2) and to the shear of the angular phase velocity which causes the "tilting" of the waves.

Although waves of the form (2) may owe their origin to baroclinic instability, once they are formed we assume they behave barotropically, satisfying the perturbation vorticity equation for nondivergent motions,

$$\left[\frac{\partial}{\partial\epsilon} + u_0\frac{\partial}{a\cos\phi\,\partial\lambda}\right]\nabla^2\psi' + \left[\beta - \frac{1}{a^2}\frac{\partial}{\partial\phi}\left(\frac{1}{\cos\phi}\frac{\partial u_0\cos\phi}{\partial\phi}\right)\right] \times \frac{\partial\psi'}{a\cos\phi\,\partial\lambda} = 0, \tag{3}$$

where

$$\nabla^2\psi' = \frac{1}{a^2\cos\phi}\left[\frac{1}{\cos\phi}\frac{\partial^2}{\partial\lambda^2} + \frac{\partial}{\partial\phi}\cos\phi\frac{\partial}{\partial\phi}\right]\psi'.$$

Substituting (1) in (3), and neglecting the variations of ψ' with ϕ in evaluating the vorticity $\nabla^2\psi'$, we obtain a familiar expression for the angular phase speed of the wave at each latitude

$$\mu(\phi) = \frac{u_0}{a\cos\phi} - \frac{a\cos\phi\left[\beta - \frac{1}{a^2}\frac{\partial}{\partial\phi}\left(\frac{1}{\cos\phi}\frac{\partial u_0\cos\phi}{\partial\phi}\right)\right]}{n^2}. \tag{4}$$

It now remains for us to determine the characteristic "tilting" time T. For this purpose we require that the transports given by (2) allow an equilibrium to be established between the generation of zonal kinetic energy and its dissipation. Formally, we require that (2) satisfy the mean zonal flow kinetic energy equation for the time mean flow (cf. Murakami [2]),

$$\left\{(u'v')_0\cos\phi\frac{\partial(u_0/\cos\phi)}{a\partial\phi}\right\} + \left\{(u_1v_1)_0\cos\phi\frac{\partial(u_0/\cos\phi)}{a\partial\phi} + (u_1\omega_1)_0\frac{\partial u_0}{\partial p}\right\} + \left\{\left(f + \frac{\tan\phi}{a}u_0\right)u_0v_0\right\} + \left\{(u'\omega')_0\frac{\partial u_0}{\partial p}\right\} + \{u_0x_0\} = 0. \tag{5}$$

If we assume the vertical transient eddy stresses are in the nature of an eddy viscosity, we can combine the last two terms as an overall measure of the eddy dissipation of the mean zonal flow for which we can venture the approximate form,

$$\left\{(u'\omega')_0\frac{\partial u_0}{\partial p} + u_0x_0\right\} = -C\{u_0^2\},$$

where C is an empirical constant. Thus, substituting for $(u'v')_0$ from (2), we obtain from (5),

$$T = \left[C\{u_0^2\} - \left\{(u_1v_1)_0\cos\phi\frac{\partial(u_0/\cos\phi)}{a\partial\phi} + u_1\omega_1)_0\frac{\partial u_0}{\partial p}\right\} - \left\{\left(f + \frac{\tan\phi}{a}u_0\right)u_0v_0\right\}\right] \Big/ \left\{\sigma_v^2\cos^2\phi\frac{d\mu}{d\phi}\frac{\partial(u_0/\cos\phi)}{a\partial\phi}\right\}. \tag{6}$$

This expression for T, together with the expression for μ, enables us to use (2) as a parameterization of the momentum transport in terms of the mean zonally averaged variables σ_v, u_0, and v_0, which, in the complete symmetric problem, are dependent variables (cf. Saltzman [5]). The standing eddy stresses appearing in (6) are presumed to be determined from the theory of stationary perturbations (cf., e.g., Saltzman [4]), or prescribed. For a homogeneous earth, these standing eddy terms vanish.

3. TEST RESULTS

As a test of the formula, we first evaluated (2) at 500 mb. for values of n ranging from 4 to 14, specifying the observed 6-mo. average winter and summer values of u_0, σ_v, and $(u_1v_1)_0$ all derived from data compiled recently by Starr and Frazier [8]. In evaluating T, we included the effects of the integrals involving v_0 and ω_1 (which are not yet reliably measured) by choosing C such that the balance (5), based on the *observed* field of $(u'v')_0$, is achieved without them. This value turns out to be $C=0.024$ days^{-1}. The observed values of u_0 and σ_v are shown in figure 1, and the computed and observed values of $(u'v')_0$ at 500 mb. are shown in figure 2. It can be seen that the computed values in middle and low latitudes are a good approximation to the observed for wave numbers in the range of 6, 7, and 8, which are waves containing a great deal of the total meridional transient eddy kinetic energy (cf. Saltzman and Fleisher [6]). The southward flux of momentum in polar latitudes appears to be underestimated for all n, particularly in summer. The formula does reproduce what is perhaps the most important feature of the $(u'v')_0$ distribution, namely the fluxes counter to the gradient of the relative angular momentum of the zonal flow.

In principle, the wave number dependence can be taken into account more rigorously by resolving the total σ_v used in this calculation into harmonic components and summing the contributions of all the waves. In general, for a fixed σ_v, the lower the wave number the larger the northward momentum transport in middle latitudes.

We note that one can qualitatively take into account the meridional variations of the amplitude of the waves (which were neglected in deriving (4)) by taking n as a kind of vector wave number for both zonal and meridional variability. In this case, n should take on a value higher than the purely zonal wave number.

In figure 3a, b, we show the computed and observed values of $(u'v')_0$ as a function of pressure height, for $n=6$ only. It can be seen that the formula gives reasonably

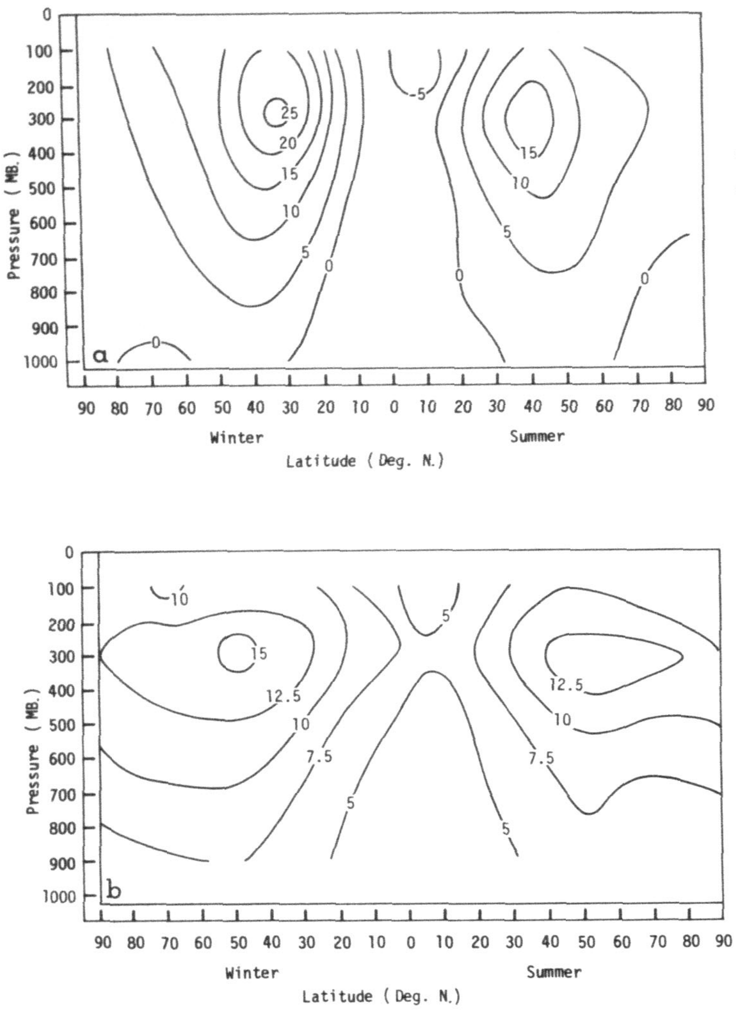

FIGURE 1.—(a) Observed 6-mo. average fields of u_0; and (b) of $\sigma_v=\sqrt{(v'^2)_0}$, derived from data compiled by Starr and Frazier [8]. Units are m. sec.$^{-1}$

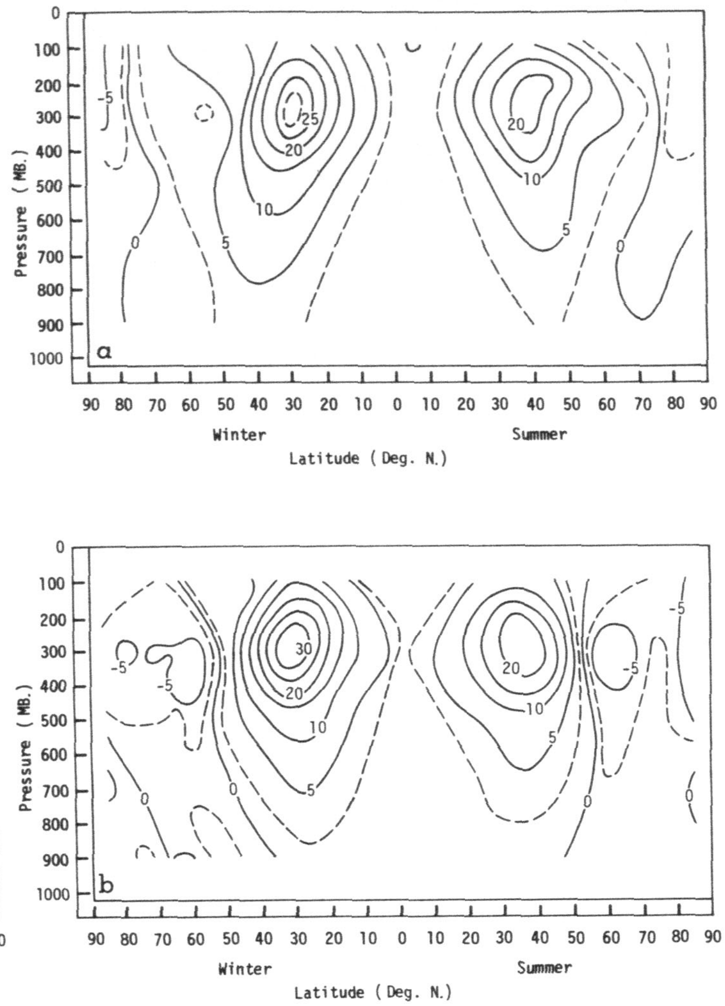

FIGURE 3.—(a) Computed values of $(u'v')_0$ as a function of pressure height, for $n=6$. (b) Observed values derived from data compiled by Starr and Frazier [8]. Units are m.^{2} sec.$^{-2}$

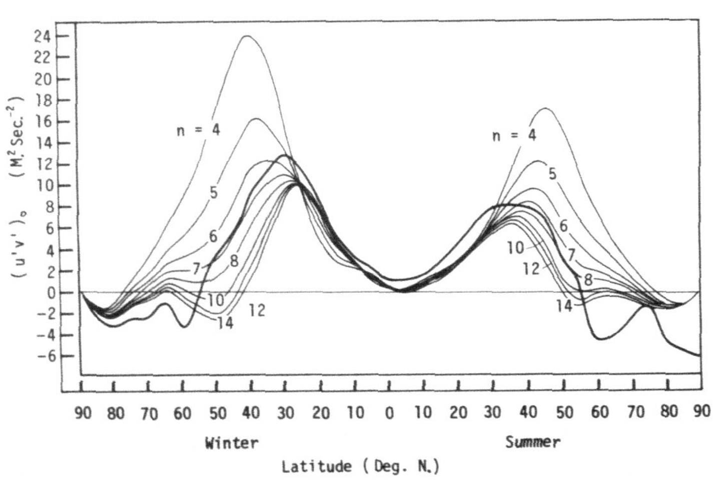

FIGURE 2.—Values of $(u'v')_0$ calculated at 500 mb. from equations (2), (4), and (6), as a function of wave number, n. Observed values, derived from data compiled by Starr and Frazier [8], are shown by the heavy line. Units are m.^{2} sec.$^{-2}$

good results for levels other than 500 mb. The value of T corresponding to $n=6$ turns out to be 0.23 days.

We now plan to use the parameterization represented by (2), (4), and (6) in a deterministic system for the mean symmetric state in which σ_v, u_0, and v_0 are free dependent variables.

NOTE ADDED IN PROOF: It has recently come to our attention that physical principles similar to those described here were previously applied to short period baratropic variations by A. Arakawa (*Journal of the Meteorological Society of Japan*, Vol. 39, No. 2, Apr. 1961, pp. 49–58).

ACKNOWLEDGMENTS

We are grateful to Prof. V. P. Starr and Mr. H. Frazier for making their as yet unpublished data available to us, to Mr. Frank Perry for his aid in making the calculations, and to Miss Peggy Atticks for typing the manuscript.

This work has been supported by the Environmental Science Service Administration under Contract No. E22-137-67(N), and by the Travelers Insurance Companies.

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[*Received March 6, 1968; revised May 14, 1968*]

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